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INCREMENTAL DEFORMATION MODEL FOR A SHELL

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A nonlinear deformation model for a shell with rigid transverse fibers is proposed. A complete system of incremental equations, a variational equation equivalent to this system, and a particular equation of virtual work are formulated. Numerical analysis of the nonlinear deformation of a spherical dome is performed using the complete equation.

1. Equations of Finite Deformation of a Shell. We consider a shell as a body in three-dimensional Cartesian space. The shell material is distributed over a small neighborhood G of a certain (*base*) surface $A \subset G$. A system of curvilinear coordinates t_J is attached to this surface in such a manner that t_1 and t_2 are internal parameters of the surface, and t_3 is the normal (*transverse*) coordinate ($t_i \in A$ and $t_3 \in [h_1, h_2]$, where h_i is a real number).

In this paper, we use the notation adopted in [1]. Variations of the deformation parameters are denoted by the symbol δ and the desired and specified increments are denoted by the symbol Δ. The capital Latin subscripts and superscripts take values 1, 2, and 3, and the lower-case Latin subscripts and superscripts take values 1 and 2. Summation is performed over repeated subscripts. The possible dependence on time is not indicated explicitly.

In the three-dimensional space above G, we define the position vector $\boldsymbol{g}(t_J \in G)$ of an arbitrary point of the shell, the position vector $\boldsymbol{a}(t_J \in A)$ of an arbitrary point on the base surface, and the local coordinate basis $\boldsymbol{a}_J(\boldsymbol{a})$ related to points on the surface and consisting of the tangent vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ and the normal vector $\boldsymbol{a}_3$.

The shell is defined by the relations $\boldsymbol{g} = \boldsymbol{a} + t_3\boldsymbol{a}_3$. The equalities $\boldsymbol{g}_J \equiv \partial_J \boldsymbol{g}$, $\boldsymbol{g}_3 = \boldsymbol{a}_3$, $\boldsymbol{g}_i = \boldsymbol{a}_i + t_3\boldsymbol{b}_i$, $\boldsymbol{a}_i \equiv \partial_i \boldsymbol{a}$, and $\boldsymbol{b}_i \equiv \partial_i \boldsymbol{a}_3$ (∂_J denotes differentiation with respect to t_J) introduce the body basis $\boldsymbol{g}_J(\boldsymbol{g})$ of the coordinate system and express this basis in terms of the contour basis $\boldsymbol{a}_J(\boldsymbol{a})$.

Deformation of the shell into a certain finite state is represented by the mapping $\boldsymbol{g} \to \boldsymbol{g}^+(\boldsymbol{g})$, $\boldsymbol{g}_J \to \boldsymbol{g}_J^+(\boldsymbol{g})$, and $\boldsymbol{g}_J^+ \equiv \partial_J \boldsymbol{g}^+$. The base surface and its basis deform together with the shell: $\boldsymbol{a} \to \boldsymbol{a}^+(\boldsymbol{a})$ and $\boldsymbol{a}_J \to \boldsymbol{a}_J^+(\boldsymbol{a})$. The local orthogonal transformation

$$\boldsymbol{a}_J^0 = \boldsymbol{\Theta} \cdot \boldsymbol{a}_J, \qquad \partial_3 \boldsymbol{\Theta} \equiv \boldsymbol{0}, \qquad \boldsymbol{\Theta} \cdot \bar{\boldsymbol{\Theta}} \equiv \boldsymbol{1} \tag{1.1}$$

with the rotation tensor $\boldsymbol{\Theta}(\boldsymbol{a})$ introduces, on the deformed surface, a *convective* basis $\boldsymbol{a}_J^0(\boldsymbol{a})$ with initial value $\boldsymbol{a}_J(\boldsymbol{a})$. Henceforth, this basis is assumed to be the determining basis for the vector spaces above G and A. Transformation (1.1) is represented in the form transposed with respect to that in [1], which is more customary in matrix calculus.

The deformed state of the shell is described by the equation

$$\boldsymbol{g}^+ = \boldsymbol{a}^+ + t_3\boldsymbol{a}_3^0. \tag{1.2}$$

This implies the equalities $\boldsymbol{g}_3^+ = \boldsymbol{a}_3^+ = \boldsymbol{a}_3^0$, which identify the deformed transverse vector with the convective vector. Generally, the vectors $\boldsymbol{g}_i^+$, $\boldsymbol{a}_i^+$, and $\boldsymbol{a}_i^0$ do not coincide with each other and the convective vector $\boldsymbol{a}_i^0$ is not tangent to the base surface.

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Equation (1.2) corresponds to the linear approximation of the body displacement field $\boldsymbol{w}(\boldsymbol{g}) \equiv \boldsymbol{g}^+ - \boldsymbol{g}$ with respect to the transverse coordinate:

$$\boldsymbol{w} = \boldsymbol{u} + t_3\boldsymbol{v}, \qquad \boldsymbol{u} \equiv \boldsymbol{a}^+ - \boldsymbol{a}, \qquad \boldsymbol{v} \equiv \boldsymbol{a}_3^0 - \boldsymbol{a}_3. \tag{1.3}$$

This relation specifies rigid-body motion of a transverse fiber of the shell with translational displacement $\boldsymbol{u}(\boldsymbol{a})$ and rotation $t_3\boldsymbol{v}(\boldsymbol{a})$.

The body strain field of the shell is defined by the vectors $\boldsymbol{w}_I(\boldsymbol{g})$:

$$\boldsymbol{w}_I \equiv \boldsymbol{g}_I^+ - \boldsymbol{\Theta}\cdot\boldsymbol{g}_I = \partial_I\boldsymbol{w} - (\boldsymbol{\Theta} - \boldsymbol{1})\cdot\boldsymbol{g}_I. \tag{1.4}$$

Using approximations (1.2) and (1.3), we arrive at the equalities

$$\boldsymbol{w}_i = \boldsymbol{u}_i + t_3\boldsymbol{v}_i, \qquad \boldsymbol{w}_3 \equiv \boldsymbol{0}, \tag{1.5}$$

which express the body field in terms of the surface vectors $\boldsymbol{u}_i(\boldsymbol{a})$ and $\boldsymbol{v}_i(\boldsymbol{a})$ of metric and flexural–torsional strains:

$$\begin{aligned} \boldsymbol{u}_i \equiv \boldsymbol{a}_i^+ - \boldsymbol{\Theta}\cdot\boldsymbol{a}_i = \partial_i\boldsymbol{u} - (\boldsymbol{\Theta} - 1)\cdot\boldsymbol{a}_i, \qquad \boldsymbol{a}_i^+ \equiv \partial_i\boldsymbol{a}^+, \\ \boldsymbol{v}_i \equiv \boldsymbol{b}_i^+ - \boldsymbol{\Theta}\cdot\boldsymbol{b}_i = \partial_i\boldsymbol{\Theta}\cdot\boldsymbol{a}_3, \qquad \boldsymbol{b}_i^+ \equiv \partial_i\boldsymbol{a}_3^0. \end{aligned} \tag{1.6}$$

Formulas (1.6) define these vectors in terms of *primary* unknowns — the displacement vector $\boldsymbol{u}(\boldsymbol{a})$ and the rotation tensor $\boldsymbol{\Theta}(\boldsymbol{a})$.

For a deformed state, the local dynamic equations

$$\nabla_i\boldsymbol{x}^i + \boldsymbol{p} = \boldsymbol{0}, \quad \nabla_i\tilde{\boldsymbol{y}}^i + \tilde{\boldsymbol{x}} + \tilde{\boldsymbol{q}} = \boldsymbol{0}, \quad \tilde{\boldsymbol{x}} \equiv \boldsymbol{a}_i^+ \times \boldsymbol{x}^i, \quad \tilde{\boldsymbol{y}}^i \equiv \boldsymbol{a}_3^0 \times \boldsymbol{y}^i \tag{1.7}$$

hold on A (∇_i denotes covariant differentiation with respect to the initial surface basis $\boldsymbol{a}_J$).

If, on a segment C_λ of the base surface, the contour forces and moments are specified by the vectors $\boldsymbol{p}_3$ and $\tilde{\boldsymbol{q}}_3$, the dynamic conditions

$$e_{3i}\boldsymbol{x}^i - \boldsymbol{p}_3 = \boldsymbol{0}, \qquad e_{3i}\tilde{\boldsymbol{y}}^i - \tilde{\boldsymbol{q}}_3 = \boldsymbol{0} \tag{1.8}$$

are satisfied on this segment. On a segment C_μ with specified displacement vector $\boldsymbol{u}_\mu$ and $\boldsymbol{\Theta}_\mu$ rotation tensor, the following kinematic conditions hold:

$$\boldsymbol{u} = \boldsymbol{u}_\mu, \qquad \boldsymbol{\Theta} = \boldsymbol{\Theta}_\mu. \tag{1.9}$$

The unknown vector functions $\boldsymbol{x}^i(\boldsymbol{a})$ and $\boldsymbol{y}^i(\boldsymbol{a})$ in Eqs. (1.7) are the mathematical moments of the stress vector $\boldsymbol{z}^i(\boldsymbol{g})$ along the transverse coordinate:

$$\boldsymbol{x}^i \equiv \int_{h_1}^{h_2} \boldsymbol{z}^i dG/dA, \qquad \boldsymbol{y}^i \equiv \int_{h_1}^{h_2} \boldsymbol{z}^i t_3 dG/dA. \tag{1.10}$$

The functions $\boldsymbol{x}^i(\boldsymbol{a})$ and $\tilde{\boldsymbol{y}}^i(\boldsymbol{a})$ have the meaning of *mechanical* forces and the moments.

In the general problem of the finite deformation of a shell, the dynamic equations (1.7) and (1.8) are combined with the kinematic equations (1.6) and (1.9) and constitutive relations between the dynamic vectors $\boldsymbol{x}^i$ and $\boldsymbol{y}^i$ and the kinematic vectors $\boldsymbol{u}_i$ and $\boldsymbol{v}_i$. A finite formulation of these relations is possible only in particular cases. An incremental formulation is more general. This naturally leads to the necessity of constructing an incremental deformation model for the shell.

2. Formulation of Local Equations of the Model. In formulating the incremental equations of deformation for the shell, we use the following variation rules for vector and tensor fields:

$$\begin{aligned} \delta\boldsymbol{\Theta} = \delta\boldsymbol{\Omega}\cdot\boldsymbol{\Theta}, \quad \delta\boldsymbol{\Omega} = \delta\boldsymbol{\omega}\times\boldsymbol{1} = \boldsymbol{1}\times\delta\boldsymbol{\omega}, \quad \delta\boldsymbol{a}_J^0 = \delta\boldsymbol{\Omega}\cdot\boldsymbol{a}_J^0 = \delta\boldsymbol{\omega}\times\boldsymbol{a}_J^0, \quad \delta\boldsymbol{a}_i^+ = \partial_i\delta\boldsymbol{u}, \\ \delta_0\boldsymbol{u}_i = \partial_i\delta\boldsymbol{u} - \delta\boldsymbol{\Omega}\cdot\boldsymbol{a}_i^+ = \partial_i\delta\boldsymbol{u} - \delta\boldsymbol{\omega}\times\boldsymbol{a}_i^+, \quad \delta_0\boldsymbol{v}_i = \partial_i\delta\boldsymbol{\Omega}\cdot\boldsymbol{a}_3^0 = \partial_i\delta\boldsymbol{\omega}\times\boldsymbol{a}_3^0, \quad \delta\boldsymbol{v} = \delta\boldsymbol{\omega}\times\boldsymbol{a}_3^0. \end{aligned} \tag{2.1}$$

Here $\delta\boldsymbol{\Omega}(\boldsymbol{a})$ and $\delta\boldsymbol{\omega}(\boldsymbol{a})$ are the spin and vector of virtual rotation, and δ_0 is the relative variation operator such that $\delta_0\boldsymbol{a}_J^0 \equiv \boldsymbol{0}$ and for any vector $\boldsymbol{v}$ specified in the convective basis, the equalities

$$\delta\boldsymbol{v} = \delta_0\boldsymbol{v} + \delta\boldsymbol{\Omega}\cdot\boldsymbol{v} = \delta_0\boldsymbol{v} + \delta\boldsymbol{\omega}\times\boldsymbol{v} \tag{2.2}$$

hold by definition. Formulas (2.1) contain two primary virtual vectors $\delta\boldsymbol{u}$ and $\delta\boldsymbol{v}$. The second vector has only two components in the convective basis since $\boldsymbol{a}_3^0 \cdot \delta\boldsymbol{v} \equiv 0$. The last equality in (2.1) expresses this vector in terms of the free vector of virtual rotation. In order that this expression be one-to-one, it suffices to eliminate the "drilling" component of the vector $\delta\boldsymbol{\omega}$ by the trivial condition $\delta\omega_3 \equiv \delta\boldsymbol{\omega} \cdot \boldsymbol{a}_3^0 \equiv 0$. Therefore, the three-component vector $\delta\boldsymbol{u}$ and the two-component vector $\delta\boldsymbol{\omega}$ can be regarded as the primary unknowns. In vector products, the second vector can be replaced by the spin tensor $\delta\boldsymbol{\Omega}$.

Using (1.7), we obtain the following incremental dynamic equations on A:

$$\nabla_i \Delta\boldsymbol{x}^i + \Delta\boldsymbol{p} = \boldsymbol{0}, \qquad \nabla_i \Delta\tilde{\boldsymbol{y}}^i + \Delta\tilde{\boldsymbol{x}} + \Delta\tilde{\boldsymbol{q}} = \boldsymbol{0}. \tag{2.3}$$

These are supplemented by the following dynamic and kinematic conditions on the contour segments C_λ and C_μ:

$$e_{3i}\Delta\boldsymbol{x}^i - \Delta\boldsymbol{p}_3 = \boldsymbol{0}, \qquad e_{3i}\Delta\tilde{\boldsymbol{y}}^i - \Delta\tilde{\boldsymbol{q}}_3 = \boldsymbol{0}; \tag{2.4}$$

$$\Delta\boldsymbol{u} = \Delta\boldsymbol{u}_\mu, \qquad \Delta\omega = \Delta\omega_\mu. \tag{2.5}$$

The dynamic variables are related to the kinematic variables by constitutive relations. For purely mechanical processes of elastic and elastoplastic deformation in the region G, these can be expressed by the equation

$$\Delta_0 \boldsymbol{z}^i = \boldsymbol{D}^{ij} \cdot \Delta_0 \boldsymbol{w}_j, \tag{2.6}$$

where Δ_0 is the relative increment operator defined similarly to δ_0 and $\boldsymbol{D}^{ij}$ are the dyadic tensors of the material stiffness that take the loading prehistory into account.

From (1.10) and (2.6) we obtain the following constitutive relations on A for the surface variables:

$$\Delta_0\boldsymbol{x}^i = \boldsymbol{E}_1^{ij} \cdot \Delta_0\boldsymbol{u}_j + \boldsymbol{E}_2^{ij} \cdot \Delta_0\boldsymbol{v}_j, \quad \Delta_0\boldsymbol{y}^i = \boldsymbol{E}_2^{ij} \cdot \Delta_0\boldsymbol{u}_j + \boldsymbol{E}_3^{ij} \cdot \Delta_0\boldsymbol{v}_j \tag{2.7}$$

with the generalized stiffness tensors

$$\boldsymbol{E}_N^{ij} \equiv \int_{h_1}^{h_2} \boldsymbol{D}^{ij} t_3^{N-1} \frac{dG}{dA}.$$

The vectors $\Delta_0\boldsymbol{u}_i$ and $\Delta_0\boldsymbol{v}_i$ in (2.7) are calculated according to the variation rules (2.1):

$$\Delta_0\boldsymbol{u}_i = \partial_i\Delta\boldsymbol{u} - \Delta\boldsymbol{\Omega}\cdot\boldsymbol{a}_i^+ = \partial_i\Delta\boldsymbol{u} - \Delta\omega \times \boldsymbol{a}_i^+, \quad \Delta_0\boldsymbol{v}_i = \partial_i\Delta\boldsymbol{\Omega}\cdot\boldsymbol{a}_3^0 = \partial_i\Delta\omega\times\boldsymbol{a}_3^0. \tag{2.8}$$

Using (2.8) and the equality $\Delta_0\tilde{\boldsymbol{y}}^i = \boldsymbol{a}_3^0\times\Delta_0\boldsymbol{y}^i$, which is valid by definition, we bring Eqs. (2.7) to the form

$$\Delta_0\boldsymbol{x}^i = \boldsymbol{E}^{ij}\cdot(\partial_j\Delta\boldsymbol{u} - \Delta\boldsymbol{\Omega}\cdot\boldsymbol{a}_j^+) + \boldsymbol{F}^{ij}\cdot\partial_j\Delta\omega, \quad \Delta_0\tilde{\boldsymbol{y}}^i = \boldsymbol{G}^{ij}\cdot(\partial_j\Delta\boldsymbol{u} - \Delta\boldsymbol{\Omega}\cdot\boldsymbol{a}_j^+) + \boldsymbol{H}^{ij}\cdot\partial_j\Delta\omega \tag{2.9}$$

with the modified stiffness tensors

$$\boldsymbol{E}^{ij} \equiv \boldsymbol{E}_1^{ij}, \quad \boldsymbol{F}^{ij} \equiv -\boldsymbol{E}_2^{ij}\times\boldsymbol{a}_3^0, \quad \boldsymbol{G}^{ij} \equiv \boldsymbol{a}_3^0\times\boldsymbol{E}_2^{ij}, \quad \boldsymbol{H}^{ij} \equiv -\boldsymbol{a}_3^0\times(\boldsymbol{E}_3^{ij}\times\boldsymbol{a}_3^0).$$

Henceforth, we assume that relations (2.9) admit the inversion

$$\partial_i\Delta\boldsymbol{u} - \Delta\boldsymbol{\Omega}\cdot\boldsymbol{a}_i^+ = \tilde{\boldsymbol{E}}_{ij}\cdot\Delta_0\boldsymbol{x}^j + \tilde{\boldsymbol{F}}_{ij}\cdot\Delta_0\tilde{\boldsymbol{y}}^j, \qquad \partial_i\Delta\omega = \tilde{\boldsymbol{G}}_{ij}\cdot\Delta_0\boldsymbol{x}^j + \tilde{\boldsymbol{H}}_{ij}\cdot\Delta_0\tilde{\boldsymbol{y}}^j \tag{2.10}$$

with the known compliance tensors $\tilde{\boldsymbol{E}}_{ij}$, $\tilde{\boldsymbol{F}}_{ij}$, $\tilde{\boldsymbol{G}}_{ij}$, and $\tilde{\boldsymbol{H}}_{ij}$.

Equations (2.3)–(2.5) and (2.9) or (2.10) form a complete system of local equations for the incremental deformation model for a shell.

3. Variational Formulation of the Problem. In the functional space $L_2(A)$, we introduce arbitrary variations $\delta\boldsymbol{u}$, $\delta\omega$, $\delta\boldsymbol{x}^i$, and $\delta\boldsymbol{y}^i$ of the kinematic and dynamic vectors. The local equations (2.3)–(2.5) and (2.10) are replaced by the Galerkin integral equality

$$\int_A ((\partial_i\Delta\boldsymbol{x}^i + \Delta\boldsymbol{p})\cdot\delta\boldsymbol{u} + (\partial_i\Delta\tilde{\boldsymbol{y}}^i + \Delta\tilde{\boldsymbol{x}} + \Delta\tilde{\boldsymbol{q}})\cdot\delta\omega)\, dA$$

$$+ \int_A (\tilde{\boldsymbol{E}}_{ij}\cdot\Delta_0\boldsymbol{x}^j + \tilde{\boldsymbol{F}}_{ij}\cdot\Delta_0\tilde{\boldsymbol{y}}^j - \partial_i\Delta\boldsymbol{u} + \Delta\boldsymbol{\Omega}\cdot\boldsymbol{a}_i^+)\cdot\delta\boldsymbol{x}^i\, dA$$

$$+\int_A (\tilde{\boldsymbol{G}}_{ij}\cdot\Delta_0\boldsymbol{x}^j+\tilde{\boldsymbol{H}}_{ij}\cdot\Delta_0\tilde{\boldsymbol{y}}^j-\partial_i\Delta\boldsymbol{\omega})\cdot\delta\tilde{\boldsymbol{y}}^i\,dA+\int_{C_\lambda}((\Delta\boldsymbol{p}_3-e_{3i}\Delta\boldsymbol{x}^i)\cdot\delta\boldsymbol{u}+(\Delta\tilde{\boldsymbol{q}}_3-e_{3i}\Delta\tilde{\boldsymbol{y}}^i)\cdot\delta\boldsymbol{\omega})\,dC$$

$$+\int_{C_\mu} e_{3i}((\Delta\boldsymbol{u}-\Delta\boldsymbol{u}_\mu)\cdot\delta\boldsymbol{x}^i+(\Delta\boldsymbol{\omega}-\Delta\boldsymbol{\omega}_\mu)\cdot\delta\tilde{\boldsymbol{y}}^i)\,dC=0. \tag{3.1}$$

After integration of the first integral by parts, equality (3.1) takes the form

$$\int_A (\Delta\boldsymbol{p}\cdot\delta\boldsymbol{u}-\Delta\boldsymbol{x}^i\cdot\partial_i\delta\boldsymbol{u}+(\Delta\tilde{\boldsymbol{q}}+\Delta\tilde{\boldsymbol{x}})\cdot\delta\boldsymbol{\omega}-\Delta\tilde{\boldsymbol{y}}^i\cdot\partial_i\delta\boldsymbol{\omega})\,dA$$

$$+\int_A (\tilde{\boldsymbol{E}}_{ij}\cdot\Delta_0\boldsymbol{x}^j+\tilde{\boldsymbol{F}}_{ij}\cdot\Delta_0\tilde{\boldsymbol{y}}^j-\partial_i\Delta\boldsymbol{u}+\Delta\boldsymbol{\Omega}\cdot\boldsymbol{a}_i^+)\cdot\delta\boldsymbol{x}^i\,dA+\int_A (\tilde{\boldsymbol{G}}_{ij}\cdot\Delta_0\boldsymbol{x}^j+\tilde{\boldsymbol{H}}_{ij}\cdot\Delta_0\tilde{\boldsymbol{y}}^j-\partial_i\Delta\boldsymbol{\omega})\cdot\delta\tilde{\boldsymbol{y}}^i\,dA$$

$$+\int_{C_\lambda}(\Delta\boldsymbol{p}_3\cdot\delta\boldsymbol{u}+\Delta\tilde{\boldsymbol{q}}_3\cdot\delta\boldsymbol{\omega})\,dC+\int_{C_\mu} e_{3i}(\Delta\boldsymbol{x}^i\cdot\delta\boldsymbol{u}+\Delta\tilde{\boldsymbol{y}}^i\cdot\delta\boldsymbol{\omega})\,dC$$

$$+\int_{C_\mu}((\Delta\boldsymbol{u}-\Delta\boldsymbol{u}_\mu)\cdot\delta\boldsymbol{x}^i+(\Delta\boldsymbol{\omega}-\Delta\boldsymbol{\omega}_\mu)\cdot\delta\tilde{\boldsymbol{y}}^i)\,dC=0. \tag{3.2}$$

This form requires the smoothness of the variation $\delta\boldsymbol{u}$ and $\delta\boldsymbol{\omega}$ above the base surface.

When the integrands are sufficiently smooth, the variational equalities (3.1) and (3.2) are equivalent and, hence, the following statement is valid.

Statement. *If the vectors $\Delta\boldsymbol{x}^i$, $\Delta\tilde{\boldsymbol{y}}^i$, $\Delta\boldsymbol{u}$, and $\Delta\boldsymbol{\omega}$ are an exact solution of the system of local equations* (2.3)–(2.5) *and* (2.10), *the integral equality* (3.2) *holds for any variations; if certain vectors $\Delta\boldsymbol{x}^i$, $\Delta\tilde{\boldsymbol{y}}^i$, $\Delta\boldsymbol{u}$, and $\Delta\boldsymbol{\omega}$ identically satisfy equality* (3.2) *for any variations, these vectors are an exact solution of the above-mentioned system.*

When the desired integrands are insufficiently smooth, the variational equation (3.2) gives a weak formulation of the incremental problem. In this Galerkin formulation, the smoothness requirements are minimal: the vectors $\Delta\boldsymbol{x}^i$, $\Delta\tilde{\boldsymbol{y}}^i$, $\delta\boldsymbol{x}^i$, and $\delta\tilde{\boldsymbol{y}}^i$ are elements of the Hilbert space $L_2(A)$ and the vectors $\Delta\boldsymbol{u}$, $\Delta\boldsymbol{\omega}$, $\delta\boldsymbol{u}$, and $\delta\boldsymbol{\omega}$ are elements of the Sobolev space $W_2^1(A)$.

An important corollary of (3.2) is the equation of the virtual work of the shell:

$$\int_A (\Delta\boldsymbol{p}\cdot\delta\boldsymbol{u}-\Delta\boldsymbol{x}^i\cdot\partial_i\delta\boldsymbol{u}+(\Delta\tilde{\boldsymbol{q}}+\Delta\tilde{\boldsymbol{x}})\cdot\delta\boldsymbol{\omega}-\Delta\tilde{\boldsymbol{y}}^i\cdot\partial_i\delta\boldsymbol{\omega})\,dA+\int_{C_\lambda}(\Delta\boldsymbol{p}_3\cdot\delta\boldsymbol{u}+\Delta\tilde{\boldsymbol{q}}_3\cdot\delta\boldsymbol{\omega})\,dC=0. \tag{3.3}$$

It is valid for kinematically possible variations $\delta\boldsymbol{u}$ and $\delta\boldsymbol{\omega}$ such that $\delta\boldsymbol{u}=\delta\boldsymbol{\omega}=\boldsymbol{0}$ on the contour C_μ and the local equations (2.10) and boundary conditions (2.5) are satisfied. Equality (3.3) gives a weak form of the dynamic equations (2.3) and the contour conditions (2.4). When the variables $\Delta\boldsymbol{x}^i$ and $\Delta\tilde{\boldsymbol{y}}^i$ are eliminated from Eq. (3.3) using equalities (2.9), Eq. (3.3) takes the meaning of a weak formulation of the problem relative to the kinematic variables $\Delta\boldsymbol{u}$ and $\Delta\boldsymbol{\omega}$ with the principal contour conditions (2.5).

The matrix formulation of the variational equation (3.2) required for numerical analysis is obtained by decomposing the desired vector function $\Delta\boldsymbol{u}$, $\Delta\boldsymbol{\omega}$, $\Delta_0\boldsymbol{x}^i$, and $\Delta_0\tilde{\boldsymbol{y}}^i$ in the convective basis $\boldsymbol{a}_J^0$:

$$\Delta\boldsymbol{u}=\boldsymbol{a}_J^0\Delta U^J,\quad \Delta\boldsymbol{\omega}=\boldsymbol{a}_J^0\Delta\Omega^J,\quad \Delta_0\boldsymbol{x}^i=\boldsymbol{a}_J^0\Delta X^{iJ},\quad \Delta_0\tilde{\boldsymbol{y}}^i=\boldsymbol{a}_J^0\Delta\tilde{Y}^{iJ}.$$

Similar decompositions are used for the variations $\delta\boldsymbol{u}$, $\delta\boldsymbol{\omega}$, $\delta\boldsymbol{x}^i$, and $\delta\tilde{\boldsymbol{y}}^i$. The increment of any vector $\boldsymbol{v}=\boldsymbol{a}_J^0V^J$ that is different from the primary vectors is calculated similarly to (2.2): $\Delta\boldsymbol{v}=\Delta_0\boldsymbol{v}+\Delta\boldsymbol{\Omega}\cdot\boldsymbol{v}$. Here $\Delta_0\boldsymbol{v}\equiv\boldsymbol{a}_J^0\Delta V^J$ is the relative increment and $\Delta\boldsymbol{\Omega}$ is the spin of the rotation vector increment.

Differentiation of the convective-basis vectors can be expressed by the transformation

$$\partial_i\boldsymbol{a}_J^0=\boldsymbol{C}_i^0\cdot\boldsymbol{a}_J^0,\qquad \boldsymbol{C}_i^0\equiv(\partial_i\boldsymbol{a}_J^0)\boldsymbol{a}_0^J$$

with the spin tensor $\boldsymbol{C}_i^0(\boldsymbol{a})$. Any vector $\boldsymbol{v}(\boldsymbol{a})$ specified in the convective basis is differentiated by the formula

$$\partial_i\boldsymbol{v}=\partial_i^0\boldsymbol{v}+\boldsymbol{C}_i^0\cdot\boldsymbol{v},\qquad \partial_i^0\boldsymbol{v}\equiv\boldsymbol{a}_J^0\partial_iV^J,$$

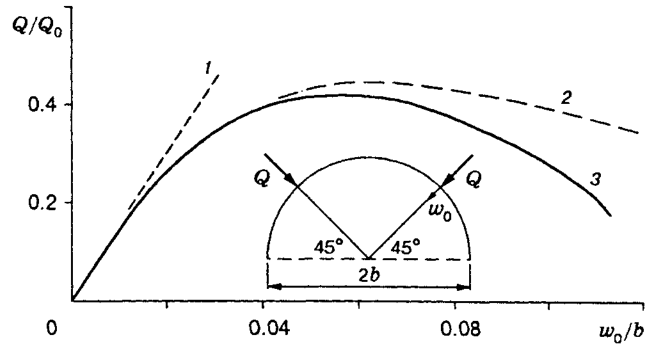

Fig. 1

where $\partial_i^0 \boldsymbol{v}$ is the relative derivative of the vector with respect to t_i. The definition of the tensor $\boldsymbol{C}_i^0$ leads to the formula $\Delta_0 \boldsymbol{C}_i^0 = \partial_i \Delta \boldsymbol{\Omega}$ for its relative increment.

The above variation and differentiation rules for vector fields are used to obtain the matrix form of the variational equation (3.2). Moreover, the relative increments and derivatives of the vectors are the basis functions in the equation since they are represented by matrices of increments of the vector components.

To calculate the desired functions, we employ the standard procedure of successive approximations, which allows one to trace the process of deformation of the shell step by step from the initial (unstressed) state to the final state corresponding to specified external forces. The initial values of the parameters are specified by the equalities $\boldsymbol{a}_J^+ = \boldsymbol{a}_J^0 = \boldsymbol{a}_J$, $\boldsymbol{b}_i^+ = \boldsymbol{b}_i \equiv \partial_i \boldsymbol{a}_3$, $\boldsymbol{C}_i^0 = \boldsymbol{C}_i \equiv (\partial_i \boldsymbol{a}_J)\boldsymbol{a}^J$, $\boldsymbol{\Theta} \equiv \boldsymbol{1}$, and $\boldsymbol{x}^i \equiv \tilde{\boldsymbol{y}}^i \equiv \boldsymbol{p} \equiv \tilde{\boldsymbol{q}} \equiv \boldsymbol{p}_3 \equiv \tilde{\boldsymbol{q}}_3 \equiv \boldsymbol{0}$. Initially, each material stiffness tensor $\boldsymbol{D}^{ij}$ is given by the general Hooke's matrix. If it depends on strains, in the next step, it is introduced by the matrix $[\boldsymbol{D}^{ij} + \Delta_0 \boldsymbol{D}^{ij}]$, where $\Delta_0 \boldsymbol{D}^{ij}$ is the relative increment of the tensor that corresponds to the increments of the primary vectors.

The body strain and stress fields in the shell are determined according the procedure outlined in [2, 3].

4. Numerical Analysis of the Deformation of a Spherical Dome. The variational equation (3.2) was used to analyze the finite elastic deformation of a spherical dome loaded by a ring force Q at an angle of 45° to the supporting plane (Fig. 1). A linear finite-element approximation of the integrands was used. Point action was specified by an U-shaped function on the cell length. The dimensions of the dome and the elastic properties of the material were specified by radius $b = 5$ m, ratio $b/h = 100$, Young's modulus $D = 10^4$ kN/m^2, and Poisson's ratio $\gamma = 0.33$ (h is the thickness of the dome). The varied parameter was the displacement w_0 of the point at which the force was applied. The magnitude of the force was calculated during solution of the problem. Axisymmetric equilibrium forms of the rigidly fixed dome were determined for successive values of the varied parameter.

The calculated nonlinear dependence of the load Q versus the displacement w_0 is shown by curve 3 in Fig. 1, where $Q_0 = Dh^2/b$. Curves 1 and 2 show the linear and quadratic approximations of the solution, respectively. One can see that the both widely used truncated models lose accuracy as the displacement w_0 increases. The calculations were carried out on an IBM PC 386.

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